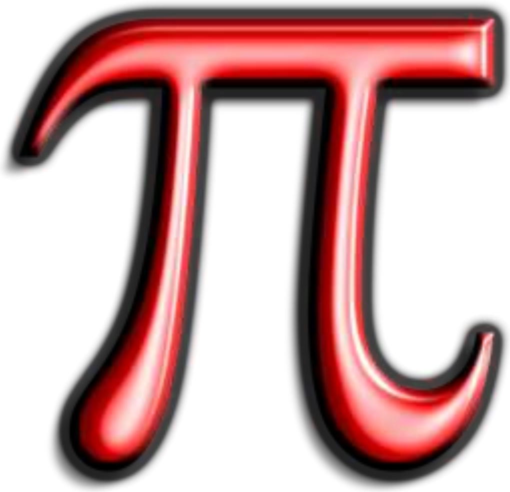
# A CONTINUED FRACTION
# TO CALCULATE THE NUMBER PI

## (An abridged translation)

$2^{nd}$ Edition



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There are many continuous fractions that produce the number Pi. I believe that the first was produced by Lord Brouncker, following a conversation with Wallis. He gave the fascinating result:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \dots}}}}$$

without telling how he had found it. Wallis took pains to demonstrate this development, raising Euler's comment that "*Wallisius*" endeavored to produce a demonstration "*quae penitus ab auctori modo diversa esse videtur (= which appears to be far from the method used by the author)*" (EULER'S INTRODUCTIO IN ANALYSIN INFINITORUM VOL. 1 Chapter 18).

If you search the Web, you frequengtly find that curious students of math ask how this beautiful formula is demonstrated, constantly receiving the answer that it follows from the *Euler theorem on the continuous fractions,*

$$a_0 + a_0a_1 + a_0a_1a_2 + \cdots + a_0a_1a_2\cdots a_n$$
$$= \cfrac{a_0}{1 - \cfrac{a_1}{1 + a_1 - \cfrac{a_2}{1 + a_2 - \cfrac{\ddots}{\ddots \cfrac{a_{n-1}}{1 + a_{n-1} - \cfrac{a_n}{1 + a_n}}}}}}$$

with the vaguely ironic assertion that this theorem can be *easily* demonstrated via mathematical induction and allows to convert a series into a continuous fraction and vice versa. Needless to say, the real problem is how to demonstrate the above-mentioned theorem, and not the slightest hint is normally given to this effect, aside from what I have just said. Mathematical induction usually means that if we can derive the term n from the term n-1, and we know that the first term is valid, then the theorem is demonstrated. Frankly, I think that a few more words might be spent on this subject telling at least what is meant by the n and n - 1 terms both in the left and in the right member. For example, whoever tries to find out what is meant by " nth term in the first member" will find that it is not the nth term of the series.

Now, while Euler in his work has the clear intention of building a general theory of continuous fractions, in the present essay I have set to myself the ridiculously modest purpose of showing (not demonstrating) how Lord Brouncker's formula can be reduced to the well known Leibniz-Gregory series. Nothing else. Those who want the most general theory have just to read the text of Euler, or more modern texts, almost all shorter - not all more immediate.

For convenience, I write a generic continuous fraction in the form:

$$z = a + \cfrac{A}{b + \cfrac{B}{c + \cfrac{C}{d + \cfrac{D}{e \ldots}}}}$$

By stopping the fraction at the first summand (the "small letter") of the nth denominator, we can reconstruct a succession of "***convergent***" fractions, which have not obvious but interesting relationships between each other. *The convergents are not, of course, the terms of a series,* but each convergent is in itself a better approximation of the number that the continued fraction, whether it has a finite number of terms or not, represents.

Convergents can be easily calculated by hand, if we are just provided with paper, pencil, and much patience. However, their calculation does not require any special trick.

There is a convergent, $C_0$, of order 0, given by 1/0. This “formal” term has been introduced by Euler just to show some regularity we will not rely upon.

The first convergent, $C_1$, which interests us is $a$. Please remember that we always have to stop at a small letter, because it is supposed that what follows is a lesser amount.

$C_2$ è dato da $a + A/b = \frac{A+ab}{b}$

$C_3$ è dato da $a + A/(b + B/c) = \frac{aB+Ac+abc}{B+bc}$

$C_4$ è dato da: $a + A/(b + B/(c + C/d) = \frac{AC+abC+aBd+Acd+abcd}{bC+Bd+bcd}$

$C_5$ è dato da: $a + A/(b + B/(c + C/(d + D/e))) = \frac{aBD+AcD+abcD+ACe+abCe+aBde+Acde+abcde}{BD+bcD+bCe+Bde+bcde}$

Besides the fact that the expressions lengthen, there is nothing new to be learned from the convergents which follow. It should be said, however, that at this point it seems that Euler amused himself to show that it is not necessary to do all the - let's admit it - tedious operations, but there is a relatively straightforward generation law for the successive convergents. If you find it without help you are good: only let me tell that it allows to construct the numerator and denominator of the convergent n using both the convergent n-1 and the convergent n-2. One of the two is not enough.

It can also be shown that, calling z the value of the continuous fraction, the various convergents are alternatively greater and smaller than z. Since they converge (not by chance they are called convergents) such differences will be smaller and smaller. Euler verifies the assumption for the first four terms, in which, by generally calling $R_i$ a positive remainder, by which we obtain the z value, which is not included in the calculation, we have respectively:

$C_0$ = 1/0, certainly greater than z;

$C_1$ = a, certainly less than z = a + $R_0$

$C_2$ = a + A / b, certainly greater than z = a + A / (b + $R_1$)

$C_3$ = a + A / ((b + B / c)), less than z = a + A / ( b + B / (c + R2) ).

(In red, we have the terms that "make a difference")

And here comes the “lion's paw”.

In general, we have, by calling the various convergents $C_n$ ,the "telescopic" series

$$z = C_1 + (C_2 - C_1) + (C_3 - C_2) + \ (C_4 - C_3) \ldots$$

which, once it is summed to the nth term, leaves us with the nth convergent, which should be more precise than all the preceding ones. This is of course always true, but it is comforting to know that the terms are always smaller because the convergents ... converge. The interest, as we said, comes from the fact that the last convergent, $C_n$, is also the sum of the first n terms of the series.

To have the different terms we just need to calculate the convergent differences as indicated, which should become smaller and smaller. But since mathematical rigor does not live on this site, we will be happy just to see that applying our formula, in the case of the number Pi, we actually find increasingly small terms in absolute value.

So we arm ourselves with patience (or with a suitable mathematics programme doing formal algebra) and we calculate the differences of the convergents, which give us the terms of the series

$$S = \ S_1 + S_2 + S_3 \ldots$$

The first difference is a - 0 = a. It will be the first term, $S_1$ of our series.

Second difference, $S_2 = \frac{A+ab}{b} - a = \frac{A}{b}$

Third difference, $S_3 = (-(A + ab)/b + \ (a + A \ / \ ((b + B \ / \ c) \ )) = \ - AB/(b \ (B + bc))$

Fourth difference, $S_4 = (a + A/(b + B/(c + C/d)) - (a + A/(b + B/c)) = \frac{ABC}{(B+bc)(bC+Bd+bcd)}$

Fifth difference, $S_5 = \ a + A/(b + B/(c + C/(d + D/e))) - (a + A/(b + B/(c + C/d))) =$

$$- \ \frac{ABCD}{(bC+Bd+bcd)(BD+bcD+bCe+Bde+bcde)}$$

The above result illustrates another property, which could be demonstrated in a rigorous way, that the $S_i$ alternate in sign.

All it remains to do is to see what happens by identifying the terms of Brouncker's continuous fraction for $\pi$ / 4, as follows:

a = 0, A = 1;

b = 1, B = 1;

c = 2, C = 9;

d = 2, D = 25;

e = 2 etc.

We get:

$S_1$= **1**

$S_2$ = - 1/1 * 3 = **-1/3**

$S_3$ = + 9/3 * (9 + 2 + 4) = **1/5**

$S_4$ = - 9 * 25 / ((9 + 2 + 4) (25 + 50 + 18 + 4 + 8)) = -9 * 25/1575 = -1/7

The series becomes:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

That is, our Gregory series.

Well, let's talk about the emotion mathematics can donate, even to amateurs! Seeing the Gregory series emerging in this way, from these absurdly large terms, I must admit that I experienced some emotion.

It is only a pity that the continued fraction of Lord Brouncker and the Gregory-Leibniz series, as we have seen, converge at the same speed. Five billion terms are required for the Gregory-Leibniz series to have Pi with 10 decimal places, from which we can deduce that we need the 5-billionth convergent of Lord Brouncker to achieve the same precision, an outrageously long fraction, albeit very elegant.