LAGRANGIAN POINTS WITHOUT EQUATIONS

Answer to a question , which appeared on Quora:

[In layman's terms, what is the concept of Lagrangian points?]

(II Edition)

I would like to make an attempt, of giving an idea of what the five Lagrangian (0) points of celestial mechanics are, without using a single formula of mathematics or physics. Should I succeed, I do not want to create any illusions. It is a chance if the main idea of the Lagrangian points can be somehow presented without hard-core mathematics, but, as Galileo said, the language of the book of Nature is mathematics: without mathematics, there can be only an approximate knowledge of Nature, based on memory and/or vague, hand-waving arguments.

For completeness, I will attach the basic equations and further explanations (at the level of senior high-school math and physics) in Part II, as notes to the text.

PART I.

I. An analogy: a battleground as a vector field (the E-field).

Let's consider a small portion of an infantry battlefield of WWI. It is "no man's land", where previous battles have been fought. On the right, there is the Blue side, with a line of hidden machine guns, indicated as short blue lines (M). The White side, unaware of the machine guns, advances from left to right, abandoning its secure line, which is on A. On the battleground, which is supposed to be basically flat, I have put one bunker (B) belonging to the Whites, two mounds (C, D), one bomb crater (E), one abandoned old trench T in front of the machine guns. Once the Whites arrive well beyond the trench, close to the machine gun line, the machine guns open the fire.

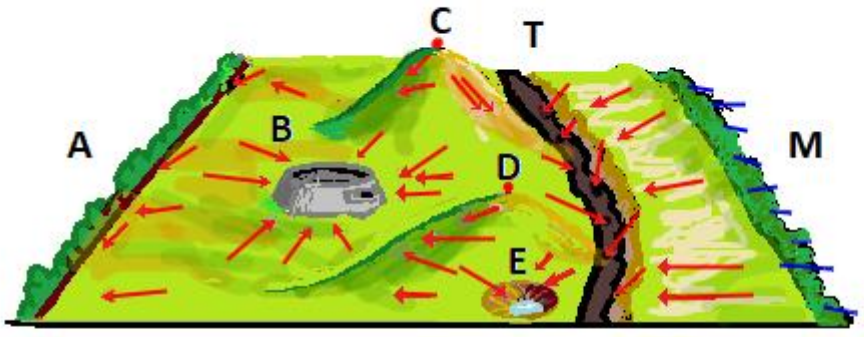

Fig.1

No man's land in WWI. The distance from the line A to M is about 500 m. The Whites advance from left to right, unaware of the machine gun line M. When the first line of the Whites is at less than 100 m from M, the machine guns open fire. The Whites run for cover.

At this point the Whites run for cover. We can put on the map little arrows indicating what a White soldier would do if he were precisely at the point where we draw the tail of the arrow. He might run in any direction, but soon a pattern develops, and the length of the arrow indicates how fast he would run. He would definitely run away from the exposed positions, especially those in front of the machine guns. If the soldier is on the first line, between the trench and the machine guns, no doubt he would run to the trench, hoping to get to the friendly line later. Most probably, if given a chance, he would head for the bunker B as the safest place (generally a bunker is the entrance of an underground structure, and we hope that the Whites had enough sense to put the entrance to the bunker facing their lines,) and could stay there permanently. The soldier might also go into the bomb crater E, but would try to get away, because the Blue could call in mortars, and the place could become dangerous.

We will call the plot of all arrows the **"ESCAPE field" or E-Field.** If a physicist were to look at the diagram without knowing what it refers to (imagine a map with the arrows only, without all the little drawings in color to explain what is it about,) he would just see one arrow drawn at each point and would say that the diagram represents a "**vector field**". That's the least he could say. In classical mechanics, indeed, since the mid-XIX century, vectors are represented by arrows. *Vector* is a generic enough word to indicate *anything, which has a size and a direction.* Displacements are vectors, velocities are vectors, accelerations are vectors, forces are vectors, and many other physical variables are vectors. Drawing all the arrows, soon makes a mess of the diagram, also because on a plane, a vector has two (positive or negative) components, in space has three, like in Fig. 2. In the present exercise, we will confine ourselves to 2D fields.

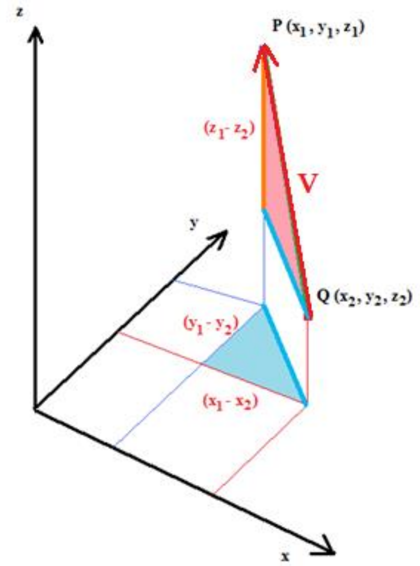

Fig.2:
A vector is drawn as an arrow, and has three components in a 3D space (red triangle), and two in a 2D plane, of the form x(1)-x(2), y(1)-y(2), [z(1)-z(2)].

Now we can ask: is there a method in the E-diagram? In which direction will a soldier run from a given point? On the spot, many thoughts, feelings, instinctive urges, etc. could lead to a more or less unreasonable choice, but a short reflection reveals that *the dominant, objective, common to all, is to escape from a "high danger "position to one of "low danger."* It is like water, which, left to itself, also goes from high altitude positions to low altitude ones. Or like a bead on a non-flat surface, and subject to the force of gravity. Or like a positive electric positive charge, which goes from a high electric potential point to a lower potential point. For the "simplest vector fields "("simple field" is a vague expression, which physicists have refined **(1))** one can always simplify the problem by creating a new field, the "mother" of the vector field, in which the problem of the direction and speed of "where to go", i.e., the length and direction of the escape vector, is determined by a map in which a single number , a sort of height or "altitude", is assigned to every point, and the "natural motion" will always be from points of higher altitude to points of lower altitude *by the fastest way.* Euler (1707-1783), used for such a height the Latin name of "Altitudo" (= height). Nowadays one uses the generic name of "Potential" **(2).**

II. The analogy continues**: a battleground as a "potential" field (the D-field).**

The consideration we have just made can suggest that we draw a simpler diagram. Instead of the complicated vector field, we could contrive another diagram for the same battleground, of a new field, which we call a "**DANGER" field or D-field**. *Each point of the battleground is then assigned a height (on some appropriate scale) proportional to the "danger," which threatens the White soldier if he stays there*, and we assume that soldiers, as I just wrote, naturally run away from regions of high danger into regions of low danger. The danger is very high close to the machine guns (M) and on the two elevated positions (C, D), but behind C and D the danger drops to low values. The trench (T) has low values of danger and would show in our "D-field" diagram as a deep concavity. The same would be true for the bomb crater (E). The bunker (B) could appear like a bottomless hole, the point of lowest danger. We could call it an "attraction" point.

Drawing the new field is quite an advantage, because in the D-field we assign to each point a single value, the altitude, while in the E-field we needed two or three values for each point, that is the values of the two or three components of the vector associated to that point (fig.2), in two dimensions and three dimensions respectively. For a planar vector field, instead of the intricate pattern of bidimensional arrows, we could build a neat landscape, with (abstract) hills and depressions, and it would be easy to deduce in which direction and how fast would/should a soldier run if he were at any point. One should just look how steep is the hole or the hill and look at the maximum slope. Notice that soldiers, who at the opening of the fire are on the top of one of the two high points (C, D) in front of the machine gun have two main options: either to run behind the small mound or to jump into the trench. Behind the mound, we thus have the two small ***c*** and ***d*** "depressions," of lower danger. If a White soldier is caught at the start of the machine gun fire on the top of one of the two high points, then he knows that he must get away as fast as possible, but he might be undecided in what direction to go and might be stuck there for a few fatal seconds. We could call the two red dots on the top of mounds C and D two "unstable equilibrium" points.

It is immediate to observe that what matters are only the differences in elevation, not the elevations themselves. Our battlefield could be a high plateau at 2000 m, as it frequently was on the Italian-Austrian front in WWI, or just above the sea level, as it was in the Flanders during the same war.

A mathematician would conclude, rather pompously, that "the Potential is always defined up to an additive constant".

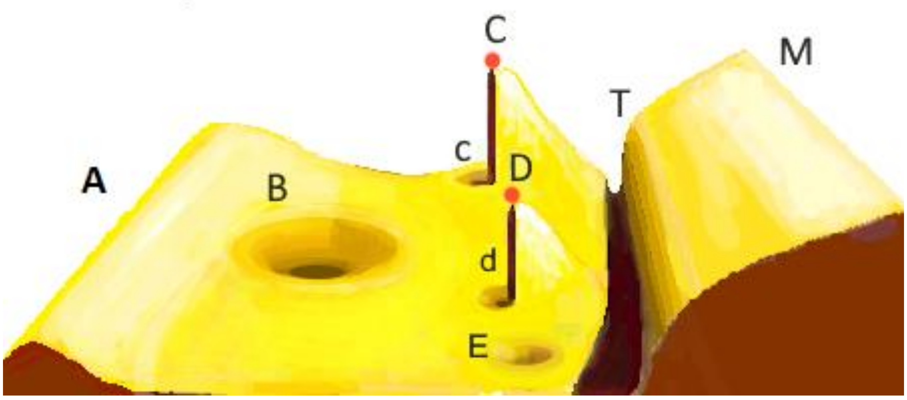

Fig.3

The D-field related to the E-field of figure 1

The discontinuities behind the mounds C and D are visible, as well the low danger c, d zones: as soon as a soldier takes cover behind the mound, he is safe form the M fire.

A mathematician could study the diagram abstractly and give directions to the White soldiers, by yelling or in some other way. But, as I said, what matters is that you can easily go from the **D- field**, a map in which each point is assigned an elevation, to the **E- field**, in which to each point an arrow is assigned. Even better: suppose you make a plaster 3D model of your D-field. Then, if you put a bead at a position on the 3D-model, it will slide into the regions of lower danger, as a White soldier should do. For example, a bead in C or D would be there in unstable equilibrium, just like a soldier not knowing where to run, into c or to T.

III. Potential, more in general.

Of course, soldiers might have various reasons, right or wrong, not to follow the "ideal" arrows and not to go to the most obvious lower danger places. So be it. We abandon our soldiers to their destiny, wishing them good luck, and we move to the rarefied heaven of classical mechanics, which Monsieur de Laplace (1749-1827) called "*The paradise of mathematical sciences.*" Celestial mechanics deals with "simpler" subjects, not endowed with free will, and tries to give mathematical solutions. For example, the arrows could represent forces and, instead of the Escape field, we would have a Force field. But, as we did for the E-field, we can construct a field whose point by point slopes are the forces. Our "D-field," in classical mechanics would become a "Potential field." Deducing the vector forces from the potential is not too complicated a business. *However, we will not need even that simple calculation. The Lagrangian points we are looking for are points of equilibrium, where the total net force is zero, and consequently, the slope is zero, like at the top of mounds C and D, in the D-diagram.*

IV. The Three-body problems. The search for a general solution.

In the XIX century, two problems in analytical mechanics proved to be so difficult that all great mathematicians tried their hand at them - in vain. They were:

1) Finding the general solution for the motion of a free **rigid body.** A general solution was never found. There are complete solutions only for a handful of particular cases. But the rigid body motion does not concern us here.

2) Finding the orbits of three mutually attracting bodies under the universal gravitation law, used by Newton (the **Three-body problem**.) A general solution was never found **(3).** It appears that long term orbits have not been found by Nature itself, and possibly do not exist. However, numerical methods can help in the short term.

Ironically, one century and a half ago the two problems presented a merely academic interest and no obvious inter-relationship. However, with the advent of the space age, satellites were launched, and both problems became important in the same context. Satellites , which were supposed to carry out long observations of the Earth , or the Sun, or the Moon, were naturally in the so-called halo orbits near equilibrium points. The attitude control of satellites is ruled by rigid body dynamics. An exhaustive discussion of the subject of artificial satellites around Lagrangian points is given in https://en.wikipedia.org/wiki/Halo_orbit; the subject of attitude control is presented in https://en.wikipedia.org/wiki/Attitude_control.

In despair of finding general solutions, mathematicians (and I mean first-class mathematicians, such as Euler (1707-1783), Laplace (1749-1827), Lagrange (1736-1813),) seeing that they had no way to find a general solution, tried to look at simplified problems and see if solutions existed for such particular cases. Euler found that there was a solution in which three bodies on a line would always rigidly rotate around their center of gravity. But the condition was that one body had to have a negligible mass with respect to the mass of the two other bodies. Such a condition allowed to separate the motion of the two larger bodies, which were not perturbed by the presence of the third body, and then find the equilibrium points for the third body. Not much later, Lagrange found that under the same conditions (one mass much smaller than the other two), there were two additional unexpected equilibrium points outside the line joining the two larger bodies. One could place satellites there and - in principle at least – the whole system would rigidly revolve as in Fig.3, or, better, as in the demonstration you find in https://en.wikipedia.org/wiki/Lagrangian_point. *We can say that the five lagrangian points are equilibrium points of the Three-body problem, in a rotating system, produced by the revolution of two main bodies around each other under the effect of the gravitational force.*

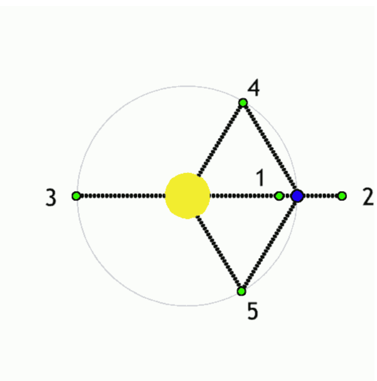

Fig.4

The five Lagrangian points (Sun(yellow) -Earth (blue) system).
Anynobody [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0)]
Fig.4 is the start of an animation to be found on Wikipedia, English edition.

As I explained, one can represent the field of forces by drawing a swarm of arrows. If one did this, one would find that at the five specific points the total forces are null, which means that a body placed there would remain there.

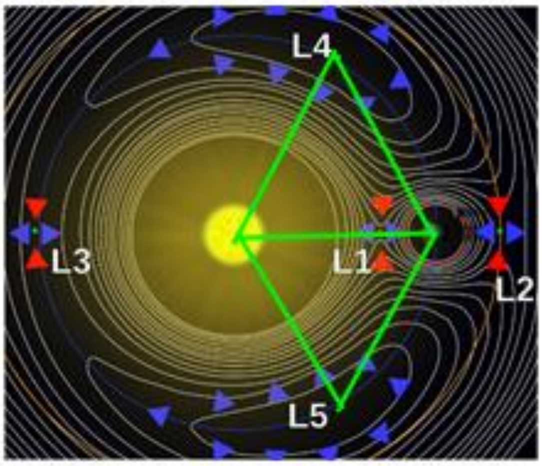

Fig. 5:
The five Lagrangian points: The force field (blue and red arrows: red arrows tend to stabilize the point, blue arrows destabilize it) superimposed to a contour plot of the effective potential, which we will deduce. I have added the two equilateral triangles, which I will mention later in the text.

How can we have an idea of how to find out that there are indeed 5 points and they are in the position indicated by the diagram? Drawing the force field would be a mess. Fig.5 just shows the situation surrounding the Lagrangian points but does not guarantee that there are no other Lagrangian points. Luckily, Fig.5 shows a contour plot of the Potential field, in which – with some exercise - it is easy to see where the equilibrium points are. I will show how to draw such a potential field and what kind of 3D model would result, which will show without problems the points where the slope is zero, and the bead placed on the plaster model would not move unless solicited by some perturbation.

I selected the most straightforward example: what happens in the particular case of two large **equal** masses, and a third minimal mass. The smallness of the third mass is the only condition imposed by Euler and Lagrange. Therefore, we expect that five points also exist in the selected case, and I can show what lies behind an accurate solution.

To have a picture of the "two masses " situation, I will consider first of all a "single mass". Like I said for the bunker in our battlefield, the attractive force by one large mass is represented by a deep funnel (in principle infinitely deep). The funnel has a profile, which is dictated by the **law of force** (4). If one wants to have an idea of what the "profile" of the funnel due to the gravitational force is, one can first have a look at a diagram designed according to the rules in Fig. 5.

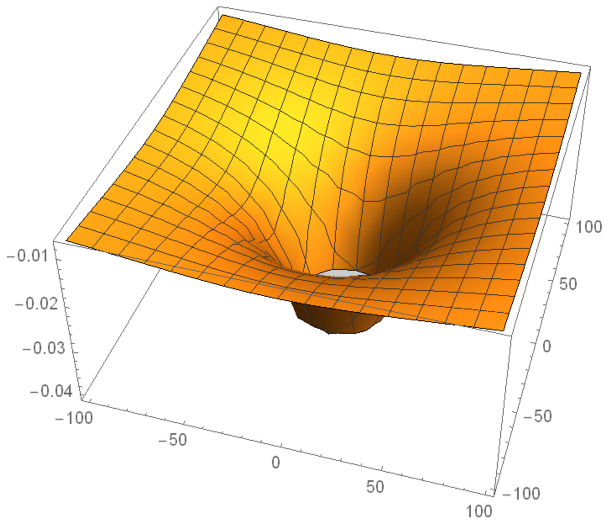

Fig.6
The gravitational field produced by one mass

Having had a look at Figure 6, one can look at a nice demonstration:

https://www.youtube.com/watch?v=Q4sh8wY3ESE

The demonstration has a value in itself, because, if you ask any non-physicist, he will tell you that a ball released at the rim of the funnel will just fall straight into it, while in a **gravitational field** we expect Keplerian orbits (first law of Kepler (1609): *The orbit of a planet is an ellipse with the Sun at one of the two foci.*) The answer of the non-physicist is correct only if the bead has no initial speed. However, if the ball has an initial velocity, not aimed straight at the source mass, the Keplerian orbits will appear. Moreover, they would go on forever: if the demonstration ends, it is merely because there is friction, which slows down the motion of the bead and complicates matters. There is a very profound reason why, but for friction, the ball should orbit indefinitely without falling to the bottom of the pit, while orbiting dangerously close to it: *its "angular momentum" must be conserved.* Why? Unfortunately, any consideration dealing with **angular momentum** tends to be rather complicated, and I leave a mere hint of it in a footnote **(5).**

Leaving aside a complex topic, which I will not mention again, let's now go back to the situation in which we place two masses on a plane. If the two masses are close enough, we create the situation of Fig.7.

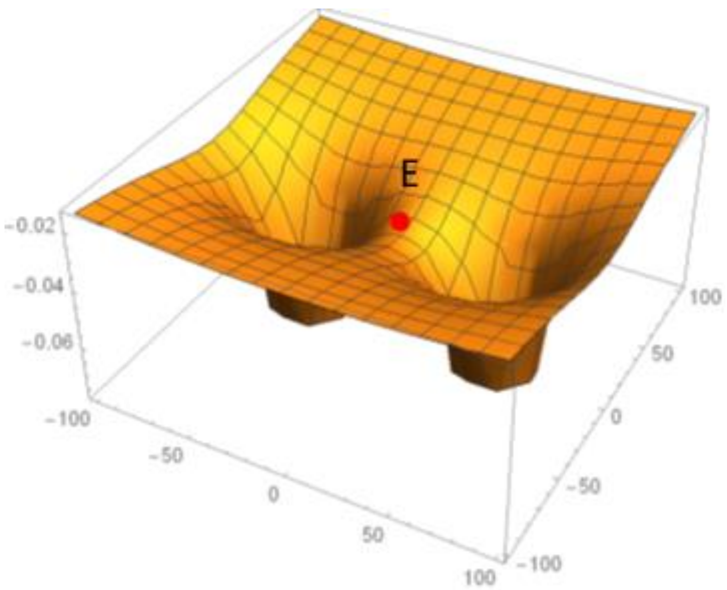

Fig.7
The gravitational field generated by two equal masses. An unstable equilibrium point (E) appears.

A test mass, a bead starting from almost any point would fall in one of the two funnels, with one exception: we see that between the two funnels, there is a point with a horizontal tangent and zero slopes, that is, *an equilibrium point*. Such a point, which is a minimum in one direction and a maximum in the orthogonal direction is an unstable equilibrium point and, as its shape suggests, is called a **"saddle point."** It is an unstable equilibrium point, in the sense that you have a problem to place a real bead there, because it tends to fall in one of the two funnels. But you can succeed, provided you keep the system perfectly motionless and don't touch the bead anymore. Not surprisingly, provided the reader believes our diagrams, while there is no equilibrium point in the whole plane, if a single gravitating mass is there, we gain one equilibrium point if we put two masses. One could think that if we add one mass on the same line, we get two equilibrium points, and so on. However, we are not allowed to think so. *The Lagrange solution is valid only if the third mass is so small that it drives no perceptible funnel on the plane.*

Therefore, we should ask, where are the other four (Euler-)Lagrange points?

We have found only one equilibrium point because we have fixed the two masses in a fixed reference system. However, we know that two gravitational masses would attract each other. They might stick to each other and sit there, in principle forever, a rare occurrence, because they must start at rest or in a head-on collision direction. Otherwise, if the two masses are free to move, classical mechanics tells us that in general, they would revolve around their center of mass. *The simplest case (not the only possible case) is if they revolve in a circular orbit.* Thus, we should put the two masses in two fixed points, yes, but *in a rotating reference system.* Such an operation can be done, but at a cost, that of subjecting them to a new force, a pseudo-force, the centrifugal force, which is not directly due to the two masses, but to the rotating system that they have created. It is a force like the one which was experienced by people entering in the Rotor carousels in fairs of old **(6)**, and therefore it will push the test particle outwards.

It can be shown **(7)** that our plane of a fixed system of reference becomes a dome if the system rotates. The dome (which is the mathematical representation of the centrifugal force) has a particular shape, which is called "inverted paraboloid of rotation," and looks like:

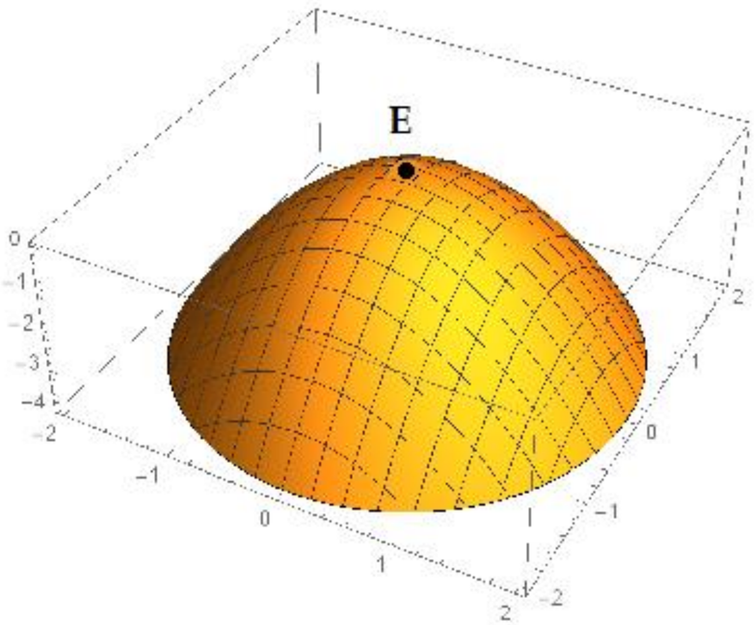

Fig. 8
The centrifugal force potential.

As we shall see, knowing the name and the precise shape of the "dome" is not dramatically important, to achieve a feeling for the five Lagrangian points.

If the third mass is very small, the circular motion of the two large masses is not perturbed. The total force acting on the small test mass is the compounded attractive gravitational force of the two masses, which pulls the particle towards the center of gravity of the system (i.e., in the present case, at midway of the two main masses,) plus a centrifugal force, which pushes the test particle away from the center of gravity of the system. *Intuitively, we expect equilibrium point(s) where the centrifugal force and the attractive force balance each other.*

The two funnels representing the two main masses should therefore be driven into the dome shaped like in Fig. 8, thus producing the following landscape (Fig.9):

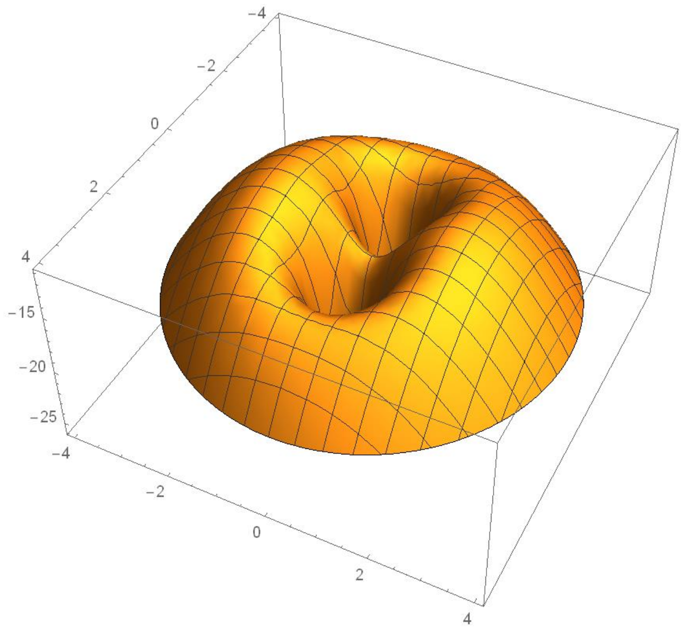

Fig.9
Two gravitational masses fixed in a rotating system. The centrifugal force appears as a dome deformed by the two gravitational funnels.

The two masses are placed in the points (x = -1, 0) and (x = +1, 0.) As the two masses are supposed to be equal, one can easily conclude that the center of gravity is at the origin (0,0), at midway between the two points.

As I anticipated, to demonstrate the existence of the five Lagrangian points, as we shall see, it is not necessary to reproduce the perfect shape of the "gravitational attraction funnels" nor of the "centrifugal dome." We will find the five Lagrangian point almost independently of the precise shape of the dome and of the funnels.

By drawing a precise as possible diagram **(8),** one obtains Fig.10:

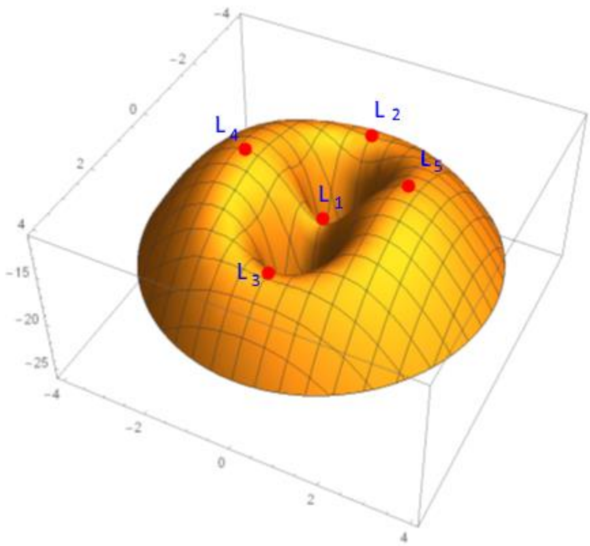

Fig.10

Same as Figure 9, but the extrema are marked. They are the 5 Lagrangian points.

Here, the five points are clearly recognizable, and their coordinates can be found by inspection, as follows:

1) The saddle point we already know between the two main masses (we call it L1), has coordinates (x = 0, y = 0);

2) Two saddle points, not as well marked as the first one, on the line joining the two masses (it would be the axis y = 0,) one at about x= -2.5, the other at the other end, x = 2.5. We call them L2 and L3. The values x= -2.5 and x = +2.5 are read out of the diagram.

3) Two broad maxima, one at coordinates x=0, y = -1.75, the other at x=0, y = +1.75. They are L4 and L5, and were the unexpected contribution of Lagrange. The values y = -1.75 and y = +1.75 are read straight out of the diagram, especially if it is oriented as in Fig.11.

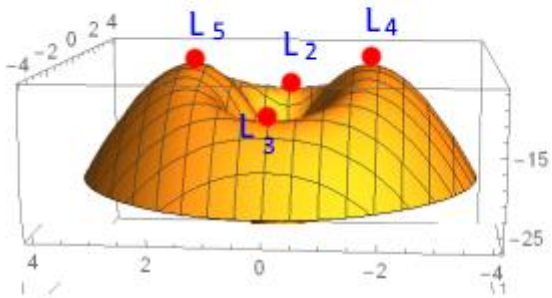

Fig. 11
A view which shows the least obvious among the Lagrangian points. The position of the two L4, L5 points at about y = 1.75 and y =-1.75. L1 is invisible in the figure.

There you have your five Lagrangian points. **At first sight, they are all points of unstable equilibrium**. A ball (a test mass) placed in one of the five Lagrangian points would appear to be in an unstable equilibrium, and at the least perturbation would fall either in one of the two pits or outside of the dome. It would appear that L4 and L5, sitting at the top of two symmetrical domes, should be the most unstable among the five points. However, going from a fixed to a rotating system we have neglected a term, *which depends on the velocity* of the test mass, and is called the "Coriolis term"**(8).**

As the velocity can be anything, we cannot make a model of the potential, which includes the Coriolis term. Thus, we have made the implicit assumption that the bead is placed at the L4 or L5 point and it stands still. The Coriolis term is not active, as the velocity is zero. As a result, the equilibrium is unstable. However, if the particle starts moving, the Coriolis term is activated, and pushes the bead in orbit close to the equilibrium point, in a mechanism similar to the formation of a cyclone: as soon as a mass of air moves into a low-pressure region, the Coriolis force makes it rotate and a fairly stable vortex is formed. Thus, the Coriolis term ensures stability, but it can be shown that it works only if the ratio of masses is *M(larger)/M(smaller) > 25*, or, more precisely **(10):**

$$\frac{M(larger)}{M(smaller)} \geq \frac{25 + 3\sqrt{69}}{2} = 24.9199$$

It follows that in the simple case I examined, with two *equal* masses, also ***L4 and L5 are unstable equilibrium points,*** like L1, L2, L3. On the other hand, in the case of the Earth and the Sun, the mass of the Earth is 3.003 10^(-6) times the solar mass and L4, L5 are stable equilibrium points. *Paradoxically, the Coriolis force transforms into two points of stable equilibrium the two Lagrangian points which appear to be the points of the least stable equilibrium***,** because they are on the top of a dome and therefore unstable in all directions, differently from L1, L2, L3, three saddle points.

Given the crudity of the plot, the values +1.75 and -1.75 for the y-coordinates of L4, L5 are safe approximations. However, should one perform precise calculations **(9),** (which amount to finding the maxima, minima and saddle points of the surface) one would find that the values are +1.73 and -1.73, that is, plus and minus the square root of three. Which means that each of the points L4 and L5 is at a distance √(1+3) = 2 from the two principal masses, which, in turn, are at a distance =2 from each other**. In other words, L4 and L5 form two equilateral triangles with the two main masses (Fig.5).**

The case of two large equal rotating masses in a circular orbit around their center of gravity is not exceptional. The Lagrange conditions only require that the third mass is negligible, when compared to the two large masses, but impose no limits on the relative sizes of the two larger masses. *In all cases which respect Lagrange's condition, there are five Lagrangian points, and, what is more,* ***(M1, M2, L4) and (M1, M2, L5) form two equilateral triangles*** **(Fig.5.).**

I have found that very few people have a clear idea of how the Lagrangian points emerge from celestial mechanics. It is funny to think that they can emerge from your kitchen: if you take some *uncooked*, symmetrical round bun and stick two fingers vertically in the dough, close to the summit, making two deep and broad enough holes (they must be deep enough to form the saddle point between them) you have a ready-made model of the five Lagrangian points. It is even funnier to consider that the model makes physical sense because if you put a small bead on it, it will have its five unstable equilibrium points like a test mass in the field of two equal (or also unequal) masses.

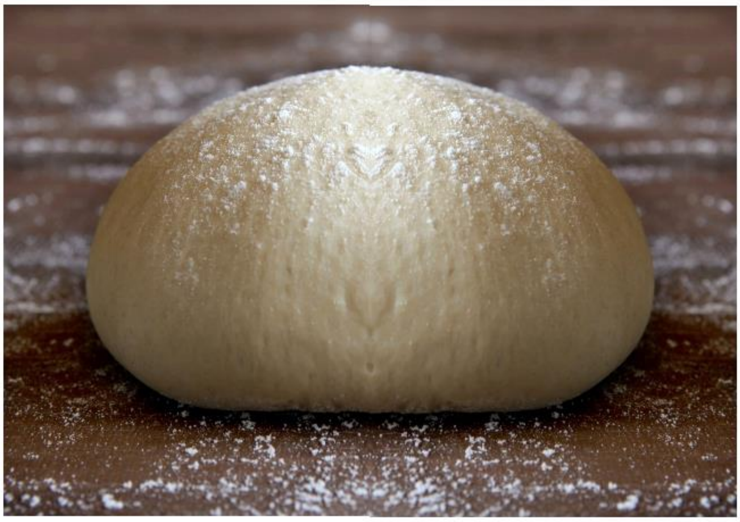

Fig.12
A round uncooked bun, ready to demonstrate the five Lagrangian points.

PART II

NOTES (Senior High School Level).

(0) The name of the Lagrangian points.

The name comes from Joseph Louis Lagrange (also Lagrangia, Lagrange, de Lagrange, as his paternal great-grandfather was French,) born in 1736 in Turin (Italy,) then capital of the kingdom of Sardinia. He left his birth city at the age of thirty, stayed 21 years in Berlin as president of the Prussian Academy of Sciences, a worthy successor of Euler. In 1787 he moved to Paris and remained in France until the end of his life. By 1802 his birth city had fallen under the French Napoleonic rule, and he became French. Lagrange was one of the top mathematicians of the end of the XVIII century and made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

(1) A "simple field" is a vague expression, which physicists have refined.

They have also invented an appropriate name: a "simple" field is a "conservative" (and "irrotational") field. Some examples are given in note (2). Fully understanding the conservative field notion requires some notion of integral calculus. If you want to know more, please look at https://en.wikipedia.org/wiki/Conservative_vector_field.

(2) Some refinements on *altitudo* and potential.

At the end of his life, Galileo had proved, after the publication of his book ***Discourses and Mathematical Demonstrations Relating to Two New Sciences*** (*Discorsi e dimostrazioni matematiche intorno a due nuove scienze,*) that a falling body, not subjected to any friction force, descending equal differences in height always achieves equal speeds. For Euler, therefore, differences in elevation were transformed into "*vis viva,*" *living force*, i.e., a quantity proportional to the square of the speed, independently of the descent path. Thus, he introduced his "*Altitudo*" (like Newton, he wrote good Latin,) or *height*, as something which "*potentially*" could become speed (squared).

Suppose we have some profile of a "mountain", the blue line in Fig.13.

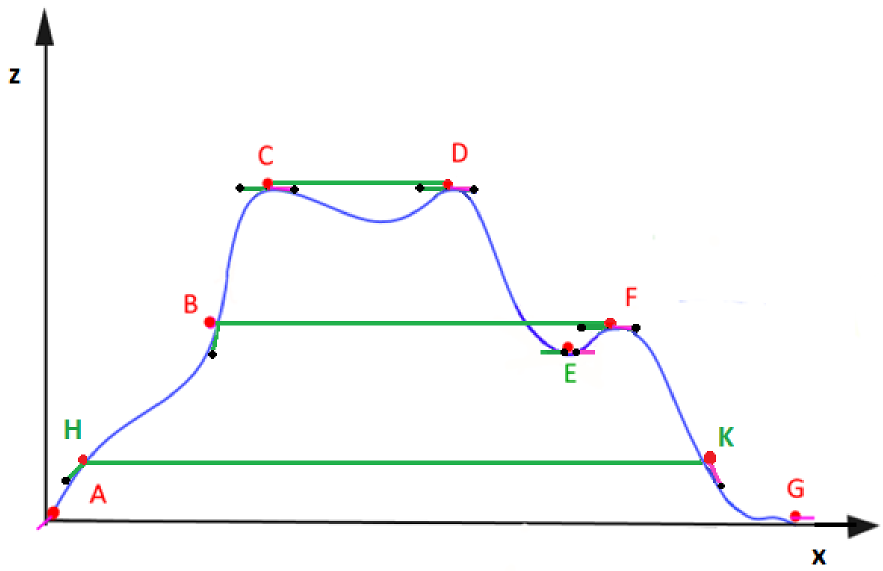

Fig.13.
"Altitudo"(=height) and applications.

The above figure tells a number of things:

(i) At the points A, B, C, D, E, F, G, H, K I have marked (indicatively) the forces, purple toward the right, green towards the left. The small black ball indicates where the force is pointing. The forces are along the tangent of the mountain profile, the blue line.

(ii) Beads in C, D, E, F are subjected to equal and opposite forces. Thus the bead is in equilibrium. However, the situation is not the same in all equilibrium points. In C, D, F, the forces tend to take the bead away from its position (the black dot is pointing away from the equilibrium point). In E it is the opposite. Thus, beads placed in C, D, F will tend to leave the equilibrium position, which therefore is an "unstable equilibrium position." E, on the other hand, is a stable equilibrium position. All tallies with our intuition. If we try to keep a bead in C, D, F, it will be difficult, in E it will be easy. You notice that at **all** the equilibrium points the tangent to the mountain profile is horizontal.

(iii) Suppose a bead is at C. It has the same elevation as D. If we release it, slightly nudging it toward the right, it will fall and then climb precisely to the same height at D – in the absence of friction. (The remark is due to Galileo, who appears to have had an exceptional talent to eliminate friction from his vision, although friction is the most ubiquitous force, *and one without which we could not live*.)

(iv) Going from C to B, the bead would acquire the same speed as a bead going from D to F.

(v) Going from B to H, a bead would acquire the same speed as going from F to K; and the same as going from C to B and from D to F.

(vi) Following Galileo, we must believe that if on the right of the diagram there was a similar mountain, a bead which started from D towards right should be able to climb to the top of that mountain.

However, a few comments are needed:

A) If we want to plot the **forces** acting on the beads at the various points of the mountain profile, we must plot, in this case, two components, one vertical and one horizontal. Either we put arrows everywhere, or we make two diagrams instead of one. If we work in three dimensions, we need three diagrams. There is a remarkable economy of thought in drawing a single mountain, and then calculating the slopes in three directions: you do it when you need it. The slopes have their mathematical names: they are called, in certain conditions, derivatives or gradients, and, given the surface of a "mountain", there are rules to draw them, and calculate their values**. In our layman approach, we will just use the easily identifiable cases of the equilibrium points:** here the slope is null, as Fig. 13 shows.

B) Euler remarked quite soon that knowing the difference in elevation of two points A and B on a "mountain" was enough to know the amount of speed a falling body would acquire (or lose, according to which point has the higher elevation) while going from A to B by any path (which is remarkable.) The formula is well known:

$$(difference\ in\ height =)\ h\ = \frac{v^2}{2g}$$

For any system of forces (be they gravitational, electrical, elastic) such that similar equations can be written,

Force = slope of a "mountain "

$$V_2 - V_1 = Kv^2$$

subjected to certain conditions**,** we can produce a *frictionless* mountain landscape, called "potential function," which tells us in which direction and with which acceleration a bead would roll (or, better, slide) and where it could stay in equilibrium. At the same time, it would give us the motion and the equilibrium points of particles subjected to the same forces, provided their mass or charge is so small that they do not deform the landscape. Because of this possibility of drawing a mountain landscape to represent the "potential" due to a given force or system of forces, Euler gave to the variable he found the name *altitudo* or "elevation" (*Mechanica,* I.3 (1736),) thus losing the privilege of being recognized as the mathematician who gave the *altitudo* the name of *potential*. The name, at least, was introduced by W. Rankine (1820-1872), but, by his times, the concept was a common notion.

(3) Note on the Three-body problem.

Heinrich Bruns (1887) and Henri Poincaré (1888-1899) finally *proved that the* "Three-body problem" *has no general solution in closed form*. Even numerical solutions are not easy to handle. Nature too, apparently, has problems with the Three-body problem, and the motion of three gravitating bodies *has no repeating orbits in the long term*, excepting specific cases.

(4) **A complement on Newton's gravitational force.**

Let's suppose that a large mass, point-like particle is placed at the origin of the plane (x, y). It will exert a gravitational force of the type which rules the gravitational attraction between point masses, that is, following the law

$$F_g = -\frac{A}{r^2}$$

where r is the distance between a test mass and the source mass, and A is a suitable constant which depends on the size of the source mass. Again, we suppose that the test mass is so small that it will not perturb the field generated by the source mass.

Why the law -A/r^2? *Newton did not know*. The "inverse square" law just produced the correct results, such as Kepler's laws. The interested reader can read the Principia, Book I, Section II, Proposition IV, Corollaria (=consequences) and Scholia (=notes.) Here Newton attributes the discovery of the law to Wren, Halley, and Hooke. Modern scholars affirm that there were also other precursors, and the inverse square law concept was rather common by the late 1670s (Newton also mentions Bullialdus and Borelli.) Besides, Newton had formulated, in Propositions 43-45 of Book I and associated sections of Book III, a sensitive test of the accuracy of the inverse square law. *He showed that only where the law of force is calculated as the inverse square of the distance will the directions of orientation of the planets' orbital ellipses stay constant as they are observed to do, apart from small effects attributable to inter-planetary perturbations* (from Wikipedia, English edition.)

For Newton, the law of universal gravitation was an empirical law found by "induction." As for himself, Newton admitted to having found no precise reason why masses should attract each other at all with the law of the inverse square law. In his 1713 edition of the *Principia*, at the end of his work, he wrote a "General Scholium" (general remark,) which contained his celebrated sentence "***hypotheses non fingo***" (I frame no hypotheses.) Later, mathematicians and physicians looked for appropriate explanations. But do we really want to know from the start more than Newton himself?

If we plot the "potential" the slope goes like -A/r (a student in mathematics would recognize that -A/r = - Integral(-A/r^2). Integration is a bit like the opposite of finding the slope.) Thus, in our landscape an attracting mass looks like a funnel placed at the origin of our plane.

(5) Angular momentum and implications.

There is a profound reason, why the ball should orbit indefinitely without falling to the bottom of the pit, while orbiting dangerously close to it: *its "angular momentum" is conserved*. Here it should be sufficient to say that **if** at the start of the fall into the funnel the speed of the ball is not zero, and/or the ball is not aimed directly at the attracting source (the center of the funnel,) the ball has an initial non-zero angular momentum, while, once it rests at the bottom of the funnel, its angular momentum is zero. The reason lies in the angular momentum expression, which is as follows: the angular momentum of the test mass m is given by the *mvr = (Mass of the test mass) by (speed at any point of the path) by (distance of closest approach to the attracting center if the particle continues along the direction of v.)* To sit on the main mass, the test mass must have lost all its angular momentum. Thus, its angular momentum has not been conserved: it has been lost, because of the action of a friction force, which, luckily, exists, but in the solar system is almost negligible. However, an inquisitive mind would ask: "Why there is such a strange law, the conservation of angular momentum?" The answer is far from immediate: it turns out that it is linked to the rotational symmetry of the physical laws, and, ultimately, to the isotropy of the Universe. Be it clear - the demonstration of the *general link between symmetries and conservation laws, is based on the so-called Noether's theorem* (from Emmy Noether, 1882-1935,) and is reserved for physics students after the third or fourth year of University.

(6) The Rotor.

Various demonstrations of the rotor-carousel or merry-go-round can be found on YouTube. The participants entering the Rotor feel pressed against the outer wall and, at a high rotation speed, might be incapable of moving toward the center of the Rotor. For a suitable rotation speed, the participants can have the impression of lying on their backs on the ground. One can easily solve the problem of the needed rotational speed: it is

$$\frac{4\pi^2}{T^2}R = g; \quad T = 2\pi\sqrt{\frac{R}{g}})$$

Where R is the rotor radius and T the rotation period; g is the acceleration of gravity.

There is an anecdote that Einstein, after observing the Rotor, came to the conclusion that gravity causes a deformation of spacetime: gravity can be simulated by a centrifugal force, originated by a rotation, which, by providing a tangential speed, if fast enough, could produce the effects of relativistic length contraction and time dilation and their consequences on length and time measurements.

(7) The centrifugal force potential.

The diagram of Fig. 7 could simply represent a hill, and knowing its precise shape is unnecessary, for the use we will make of it. However, it has been designed to represent the potential of a centrifugal force. Possibly the reader knows that there are techniques to draw 2D and 3D representation of mathematical functions. These are part of "analytical geometry," a science which was invented by René Descartes (1596-1650.) If the reader does not know, perhaps he can be contented to know that there are rules to draw representations like Fig.5 and Fig.7 of planes, curves, surfaces and more advanced mathematical objects, that the figures are mathematically rigorous, and that mathematical deductions can be drawn from them.

Using the analytical geometry technique, one could show that the "centrifugal potential" of Fig. 7 is given by "the equation of the paraboloid":

$$V = -\frac{1}{2}\,\omega^2 r^2 = -\frac{1}{2}\omega^2(x^2 + y^2)$$

where ω is the rotation speed, in radian per second. If a full revolution is completed in a time T, knowing that the value of the total angle of rotation in radians is 2π, we have the important relationship:

$$\omega = \frac{2\pi}{T}$$

Remember, we have to turn upside down the shape (as the minus sign tells us) and take the slope, thus getting an acceleration, which in this case is always in the radial direction and pushes the test mass outwards. "Calculating the slope," that is, performing the derivative with respect to r, changing sign, and multiplying by the test mass m, we obtain the law of the centrifugal force, which increases proportionally to the distance of the test mass:

$$F = m\omega^2 r$$

However, at our introductory level, such a mathematical effort is not necessary. Once we accept that a frictionless bead will be subjected to similar forces BOTH if it falls down the hill (faster and faster, driven by an increasing acceleration) AND if subjected to a centrifugal force (driven by a linearly increasing force), we are at midway.

One thing is clear: a bead situated at the top of the dome will stay there in equilibrium, with no force acting on it (at the top, the slope is zero in all direction). However, a minimal perturbation would start moving it, after which, there is no coming back. Similarly, a bead at the center of the Rotor would stand still, but a minimal perturbation would start moving it, after which, there is no coming back. We call that an **unstable equilibrium position**.

The forces acting at various points on the surface can be found by exploiting the rule "just find the maximum slopes at any point you wish, and you get the three components of the force." *Had we preferred to plot the forces directly, we would have needed three diagrams, one for each component.*

(8) The Coriolis force, 1835 (from Gaspard de Coriolis, 1792-1843.)

The Coriolis force and the centrifugal force are rather part of the acceleration in a rotating frame. They are improper forces or pseudo-forces, which need to be multiplied by the inertial mass, to assume the dimension of forces. Similarly, the gravitational force is multiplied by a mass, which is found to be exactly equal to the inertial mass, a fact, which suggested to Einstein that gravitation was also a pseudo-force of some sort.

A somewhat confusing fact results from the assumption of part of the acceleration to the left hand side of Newton's second law: the sign of the Coriolis term, as a force, is the opposite of the sign of the Coriolis term, as an acceleration.

The force acts on a body moving in a rotating frame. I give two examples taken from Feynman *(Lectures on Physics, I, 19.8).*

1*) **A person pulls a mass m along the diameter of a rotating disk.*** A curved motion results, tangent to the diameter when crossing the center of rotation. The Coriolis effect does not stop, as the centrifugal force effect does, when the mass is at the center of rotation.

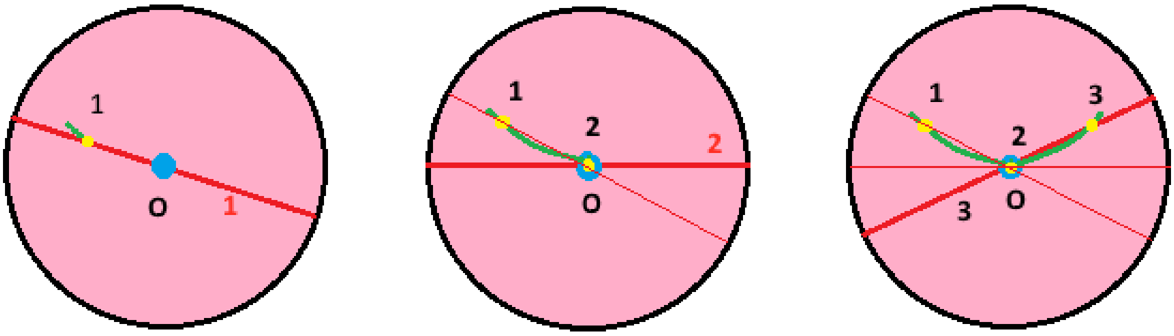

Fig. 14 (a,b,c)
The track of a person walking along a fixed diameter of a counter-clockwise rotating disk, as seen from an outside, fixed observer.

Frequently (https://en.wikipedia.org/wiki/Coriolis_force) one has the impression that the effect is exactly the opposite:

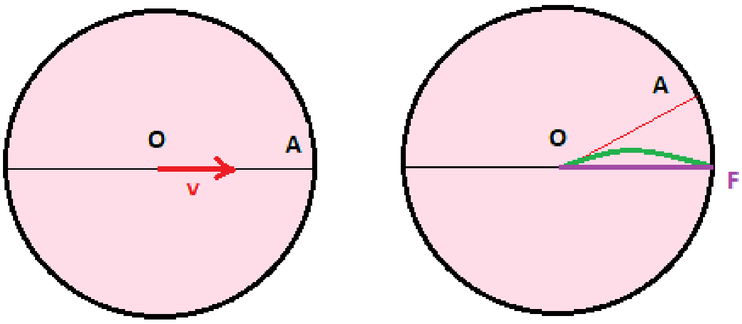

Fig. 15 (a, b)

Again, the platform is rotating counter-clockwise, but in Fig. (b) we see the path of a "flying" object, as seen by the rotating observer A, and by a fixed observer F. The fixed observer sees the straight purple colored path, the rotating observer A sees the curved "green" path.

The (in my opinion fairly confusing) difference is due to the fact that in the first case (Fig.14) the path (for example on rails) is fixed on the platform, and appears to be curved (to the East) to the external observer and straight to the rotating observer, who must, nevertheless, apply a force to have the mass m going straight on *his* path. In the second case, the path of a *flying* object, is fixed – and straight - with respect to the external observer, and appears to be curved (to its right) to the rotating observer. Most effects of the Coriolis force are seen by observers rotating with the Earth: thus, in the Northern hemisphere, rivers tend to erode their right bank, and the winds of a hurricane tend to turn in a counter-clockwise direction; the plane of oscillation of a Foucault pendulum placed at the North Pole (fixed in space) appears to rotate from east to west, etc. (See https://en.wikipedia.org/wiki/Coriolis_force).

2) Case of an observer running on the circumference of a disk.

The running observer, M, goes at a constant speed v_M.

If the disk is not rotating, the observer feels a centrifugal force, which he must counteract to follow the circumference. The fixed observer, F, sees that M must exert the same centripetal force

$$F_M = F_F = -\frac{mv_M^2}{r}$$

Such a force will continue to act if, in addition, the disk is rotating with a tangential speed ωr. M is unaware of it, but sees additional forces. For a fixed observer, the centripetal force of the running observer is

$$F = -m\frac{(v_M + \omega r)^2}{r} = -\frac{mv_M^2}{r} - 2mv_M\omega - m\,\omega^2 r$$

While the fixed observer would just consider a centripetal force in which the speed of M is $(v_M + \omega r)$, the rotating observer would consider himself subject to three forces: the first term is the centrifugal force seen by both F and M if the disk is not rotating, the last term is the centrifugal force. The middle term is the Coriolis force, which in case (1) was normal to the radial velocity, while in the present case is normal to the tangential velocity. The minus sign makes it act toward the right of v, if the rotation is in the counterclockwise direction. The force is always in the same direction, relative to the velocity, no matter in which direction the velocity is.

So far, we have considered only a planar motion and the rotation vector is normal to it.

Use of vector calculus allows one to write a more general formula for F(Coriolis):

$$F_c = -2m\,\boldsymbol{\omega} \times \boldsymbol{v}$$

Which does not depend on the distance r from the rotation center. If ω and v form an angle θ, the magnitude of the product is $2m\omega v \sin\theta$.

A nice demo of the Coriolis effect is given also in https://it.wikipedia.org/wiki/Forza_di_Coriolis

(9) A demonstration of the equilateral triangle (M1, M2, L4 or L5) for the simplest case.

It must be noticed that the masses M, their distance R, and the angular speed ω (= 2π/T as we have seen) are linked by the third Kepler law, that the "*The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit*. "

The relationship is given by

$$\frac{T^2}{R^3} = K$$

If we lengthen T (the period), and therefore we decrease the angular speed ω, we need to increase the distance R between the two masses, to preserve the above relationship.

"Trust me" (or maybe not). I made the drawing trying to ensure that the above relationship is respected. The function I plotted in 3D with the help of Mathematica (like most of diagrams in the present essay) is:

$$z = -(x^2 + y^2) - 8/\left(\sqrt{[(x+1)^2 + y^2]}\right) - 8/\left(\sqrt{[(x-1)^2 + y^2]}\right)$$

There are programs that automatically find the maximum values of functions. If we look for the maximum of the above function for **x = 0, y> 0**, we find y = 1.73205, that is the square root of 3, as announced for L4. If instead we look for the maximum in the **positive x direction**, we find x = 2.39 for L2 (instead of 2.5, as indicated approximately by the diagram). L3 is to be found at x = -2.39.

(10) A simple example to show that M1, L4, M2 and M1, L5, M2 are two equilateral triangles (Fig.5). The motion of two equal masses around their gravity center produces the third law of Kepler, which can be written as

$$\omega^2 R^3 = 2GM$$

Where R is the distance between the two large masses (in our plot R=2). Again, as the two masses are equal, the third body lies on the y axis, and its distances r from the two masses are equal.

In vector form, the gravitational force exerted by one mass should be written as

$$F_g = -\frac{GMm}{r^2}\left(\frac{r}{r}\right)$$

The term in brackets is necessary to indicate the direction of the force. As the x components of the forces due to the two masses are equal and opposite, we are only interested in the y components, which are equal, have the same sign, and are the projection of F(g) on the y axis.

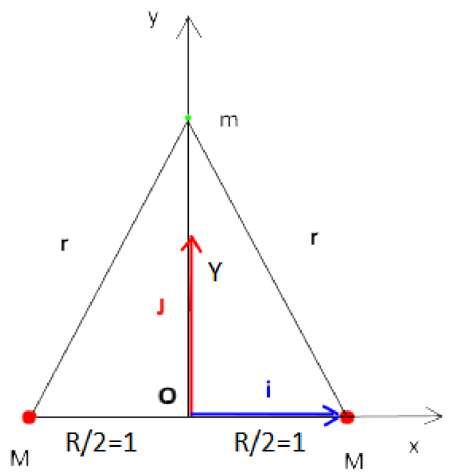

Fig.16
The case of two equal large masses.

The centrifugal force, with center in O, is totally along the y axis (unit vector j), and can be written as

$$F_c = -\frac{GMm}{R^3}\,Y$$

having used the relationship

$$\omega^2 R^3 = 2GM$$

The gravitational force acting on the mass *m* has no x-component, because the center of gravity is in the origin, at midway between the two equal main masses.

Along the y axis, the gravitational force exerted by each of the two main masses on the smaller mass is the projection (I use the internal or dot product formula. One can simply use the projection of r on the y-axis: (r . j) = r cosθ = Y:

$$F_g = -\frac{GMm}{r^2}\left(\frac{r\cdot j}{r}\right) = F_g = -\frac{GMm}{r^3}\,Y$$

$$\frac{GMm}{r^3}Y + \frac{GMm}{r^3}Y = \frac{2GMmY}{R^3}$$

Summing, simplifying 2GMmY, we have:

$$\frac{1}{r^3} = \frac{1}{R^3}$$

That is, r= R =2, and we have an equilateral triangle. In such case, the height of the equilateral triangle Y has the value Y = √(3) = 1.73205, as one can readily verify by applying Pythagora's theorem.

The general demonstration, which produces an equilateral triangle whatever is the ratio between the two main masses, is not much more difficult, and can be found in various sites online (10).

(11) A full treatment of the Lagrangian points is given in https://fr.wikipedia.org/wiki/Point_de_Lagrange (in French) as well as in other sites available on the web, also in English.

The stability is analyzed in detail in:

http://pi.math.cornell.edu/~templier/junior/final_paper/Thomas_Greenspan-Stability_of_Lagrange_points.pdf

Although fairly accessible, the stability discussions require knowledges of Calculus and Mechanics, at the level of a second/third University year.